THERMOELASTIC AXISYMMETRIC PROBLEM FOR A TWO-LAYER CYLINDER

T. G.Beleicheva and K. K. Ziling UDC 531.299:531.781

The majority of devices and units of microelectronics are multilayer structures made of materials with differing coefficients of thermal expansion and elastic constants. Thermal stresses which arise in such systems due to temperature changes when manufactured or in operation may result in a breakdown, or plastic deformation or in a change of the physical properties of materials. At the same time, due to adopted assumptions the existing design models do not describe the stressed states in real systems of finite dimensions. The designs in [1-3] are obtained on the basis of the engineering theory of beams, and in [4, 5] the obtained solution was for the infinite strip in a half-space. In the present article a right circular cylinder of radius R was used as a mathematical model which was cut by the plane z = 0 into two layers of thickness H or H* (Fig. 1). In our considerations the quantities referring to the layer 2 are distinguished by an asterisk. The cylinder deformation problem due to the temperature lowering from T_1 to T_2 was solved within the framework of the linear theory of thermoelasticity. It was assumed that the material of each layer is homogeneous and isotropic, that the temperature is independent of the coordinates, and that the coefficients of thermal expansion α and α* are independent of T. Two formulations are analyzed.

§1. For any relation between H and H* the difference methods are used to solve the problem which consists mathematically of the Duhamel–Neumann equations [6] in spherical coordinates and of the boundary conditions expressing the absence of any forces on the body outer surfaces:

$$\sigma_{\rho\rho}(R, z) = 0, \qquad \sigma^*_{\rho\rho}(R, z) = 0; \tag{1}$$

$$\sigma_{zz}(\rho, H) = 0, \qquad \sigma^*_{zz}(\rho, -H^*) = 0; \tag{2}$$

$$\tau_{\rho z}(R, z) = 0, \qquad \tau^*_{\rho z}(R, z) = 0; \tag{3}$$

$$\tau_{\rho z}(\rho, H) = 0, \qquad \tau^*_{\rho z}(\rho, -H^*) = 0. \tag{4}$$

For ρ = 0 the axisymmetric conditions are fulfilled. Moreover, the layers are rigidly fastened onto the plane z = 0

$$\sigma_{zz}(\rho, 0) = \sigma^*_{zz}(\rho, 0); \tag{5}$$

$$\tau_{\rho z}(\rho, 0) = \tau^*_{\rho z}(\rho, 0); \tag{6}$$

$$u(\rho, 0) = u^*(\rho, 0); \tag{7}$$

$$w(\rho, 0) = w^*(\rho, 0). \tag{8}$$

In the above σ_{ii} and τ_{ij} are normal or tangential stresses, respectively; u and w are the projections of the displacement on the axes of ρ and z, respectively. The state for T_1 is regarded as unstrained.

The original problem can be reduced to an equivalent variational one in which the potential energy W of the system is to be minimized. The boundary conditions for the stresses are satisfied exactly when W is minimized. An expression for W in the case of an axisymmetric strain was found in [7]. The variational problem δW = 0 written down in the continuous variables u and w is again replaced by the variational problem in the discrete values u_{ij} and w_{ij}. The employed double grid and the approximation of some of the functions and their derivatives were taken from [7]. Moreover, one uses the approximation

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 1, pp. 132-138, January-February, 1978. Original article submitted January 12, 1977.

 0021-8944/78/1901-0108$07.50

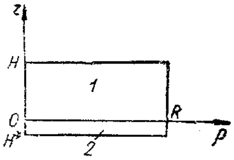

Fig. 1

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho u) \approx \frac{u_{i,j+1}-u_{i,j}}{h_1}+\frac{u_{i,j}}{\rho_j}$$

of accuracy $O(h_1)$, where the subscripts i and j denote the horizontal and the vertical grid lines ($i = 1, 2, \ldots, p$; $j = 1, 2, \ldots, q$); h_1 is the grid step length in the radius direction. The approximate expression $W(u_{i,j}, w_{i,j})$ was obtained by summing over all cells. The difference equations which approximate the original differential ones were obtained by using the conditions for stationarity values:

$$(\partial W/\partial u)_{i,j} = 0,\ (\partial W/\partial w)_{i,j} = 0.$$

In particular, the condition that $(\partial W/\partial w)_{i,j} = 0$ for a point on the separation boundary between the layers (m, j) results in the equation

$$\begin{gathered}
h_1(x^*-x)u_{m,j}+(j-1)\left(\frac{2h_1^2}{h_4}\psi^*+\frac{2h_1^2}{h_2}\psi+G^*h_4+Gh_2\right)w_{m,j}+\\
+h_1(x^*u_{m+1,j}-xu_{m-1,j})-2(j-1)h_1^2\frac{\psi^*}{h_4}w_{m+1,j}-\\
-2(j-1)h_1^2\frac{\psi}{h_2}w_{m-1,j}-\frac{h_1}{4}\left[\frac{j(1-4\nu_*)+2\nu_*}{1-2\nu_*}G^*-\frac{j(1-4\nu)+2\nu}{1-2\nu}G\right]u_{m,j+1}+\\
+\frac{G^*h_1(j-2\nu_*)}{4(1-2\nu_*)}u_{m+1,j+1}-\frac{Gh_1(j-2\nu)}{4(1-2\nu)}u_{m-1,j+1}-\frac{(2j-1)}{4}(G^*h_4+Gh_2)w_{m,j+1}+\\
+\frac{h_1}{4}\left[\frac{j(1-4\nu_*)-2(1-3\nu_*)}{1-2\nu_*}G^*-\frac{j(1-4\nu)-2(1-3\nu)}{1-2\nu}G\right]u_{m,j-1}+\\
+\left[\frac{\psi^*h_1}{2}-\frac{jG^*h_1}{4(1-2\nu_*)}\right]u_{m+1,j-1}-\left[\frac{\psi h_1}{2}-\frac{jGh_1}{4(1-2\nu)}\right]u_{m-1,j-1}-\\
-\frac{(2j-3)}{4}(G^*h_4+Gh_2)w_{m,j-1}=2(j-1)(T_2-T_1)h_1^2\left[\frac{G^*(1+\nu_*)\alpha_*}{1-2\nu_*}-\frac{G(1+\nu)\alpha}{1-2\nu}\right],
\end{gathered}$$

where

$$\psi = G(1-\nu)/(1-2\nu);\quad \psi^* = G^*(1-\nu_*)/(1-2\nu_*);$$
$$x = G\nu/(1-2\nu);\quad x^* = G^*\nu_*/(1-2\nu_*);\quad (j-1)h_1 = R.$$

In the above G is the shear modulus; ν is the Poisson coefficient. From the above equation one can also obtain the equations for interior points of the medium 1 or 2 by means of a suitable exchange of constants.

The pattern thus obtained is a nine-point pattern. All unknowns are grouped in lines as column-vectors of the type

$$\eta_j = \begin{pmatrix} u_{1,j} \\ w_{1,j} \\ u_{2,j} \\ w_{2,j} \\ \ldots \\ u_{p,j} \\ w_{p,j} \end{pmatrix}.$$

Since each η_j is related only to η_{j-1} or to η_{j+1}, therefore the matrix of the coefficients at the unknowns assumes the block-tridiagonal form. Thus, in the solution one was able to use the iterative method of linear relaxation along the lines [8] with the relaxation parameter $\varphi = 1.8$, as in [9]. This algorithm was programmed for the BESM electronic computer. In the program one took into account that for $H \ll H^*$ or for $H \gg H^*$ a considerable discrepancy between the steps h_2 and h_4 in the direction of the z axis for the media 1 and 2 can result in distorting the boundary conditions. In view of the above, equal steps were used at the boundary, the move by step taking place in the medium 1.

The values of the stress components at a point are obtained as weighted average values of these components over the area of all cells containing this point. The same approximations are used as for finite differences. In particular for the interior points of the medium 1 the stresses are evaluated by using the formulas

$$\sigma_{\rho\rho}|_{i,j} = \frac{2G}{1-2\nu}\left\{\nu\frac{u_{i,j}}{(j-1)h_1} + (1-\nu)\frac{u_{i,j+1}-u_{i,j-1}}{2h_1} + \nu\frac{w_{i-1,j}-w_{i+1,j}}{2h_2} - \right.$$
$$\left. -(1+\nu)\alpha(T_2-T_1)\right\},$$
$$\tau_{\rho z}|_{i,j} = G\left\{\frac{u_{i-1,j}-u_{i+1,j}}{2h_2} + \frac{w_{i,j+1}-w_{i,j-1}}{2h_1}\right\}.$$

Similar expressions for the medium 2 can be obtained by a suitable replacement of constants. The different expressions thus obtained approximate the differential expressions in Hooke's law with an accuracy of $O(h^2)$. For the boundary points, the formulas change and have an accuracy of $O(h)$.

In particular, on the boundary $\rho = R$ in the medium 1 one has

$$\sigma_{\rho\rho} = \frac{2G}{1-2\nu}\left\{\nu\frac{u_{i,q}}{(q-1)h_1} + (1-\nu)\frac{(u_{i,q}-u_{i,q-1})}{h_1} + \right.$$
$$\left. +\nu\frac{(w_{i-1,q}-w_{i+1,q})}{2h_2} - (1+\nu)\alpha(T_2-T_1)\right\},$$
$$\tau_{\rho z} = G\frac{(u_{i-1,q}-u_{i+1,q})}{2h_2} + G\frac{(w_{i,q}-w_{i,q-1})}{h_1}.$$

In evaluating the program a number of tests were carried out for simplified mathematical or physical models. It was thus found that for a uniform cylinder the accuracy of "zero" stresses was $\sim 10^{-4}$ kg/mm^2, the displacements agreeing with the case of free compression. The characteristic value for the accuracy of the obtained stresses was measured by the quantity

$$\delta = (|\sigma_h - \sigma_{2h}|)/|\sigma_h|,$$

where σ_h and σ_{2h} are the solutions of the difference problem obtained with the steps h_1, h_2, h_4 and $2h_1$, $2h_2$, $2h_4$, respectively.

§2. In the case of $H^* \ll H$, $H^* \ll R$ the problem can be simplified. Namely, by ignoring the flexural rigidity of layer 2, setting $\sigma_{zz}(\rho, 0) = 0$ in (5), and by replacing in (6) for that layer the effect of tangential loads by the volume forces $q(\rho)$,

$$\tau_{\rho z}(\rho, 0) = H^* q(\rho), \tag{9}$$

one obtains for $u^*(\rho)$

$$\rho^2\frac{d^2u^*(\rho)}{d\rho^2} + \rho\frac{du^*(\rho)}{d\rho} - u^*(\rho) = -q(\rho)\frac{1-\nu_*^2}{E_*}\rho^2 \tag{10}$$

with the boundary conditions given by

$$\frac{du^*(\rho)}{d\rho} - \frac{u^*(\rho)}{\rho}\Big|_{\rho=0} = 0, \quad \frac{du^*(\rho)}{d\rho} + \nu^*\frac{u^*(\rho)}{\rho}\Big|_{\rho=R} = \tag{11}$$
$$= (\alpha^* - \alpha)\,\Delta T\,(1+\nu_*)$$

(E is the Young modulus). For simplicity, it was assumed that the coefficient of heat expansion for layer 2 is $(\alpha^* - \alpha)$ and for layer 1 it is zero.

The solution of Eq. (10) under the boundary conditions (11) is given by

$$u^*(\rho) = \rho\left[(\alpha^* - \alpha)\,\Delta T + \frac{1-\nu_*^2}{2E_*}\int_\rho^R q(\rho)\,d\rho + \frac{1-\nu_*^2}{2RE_*}\int_0^R \rho^2 q(\rho)\,d\rho\right] + \frac{1-\nu_*^2}{2E_*\rho}\int_0^\rho \rho^2 q(\rho)\,d\rho.$$

If $q(\rho)$ is expanded into a series

$$q(\rho) = \sum_{k=1}^{\infty} S_k J_1(\mu_k \rho),$$

then (7) is replaced by

$$u(\rho) = \frac{1-\nu_*^2}{E_*}\sum_{k=1}^{\infty}\frac{S_k}{\mu_k^2} J_1(\mu_k\rho) + \rho\,[\alpha^* - \alpha]\,\Delta T - \frac{1-\nu_*}{E_*}\sum_{k=1}^{\infty}\frac{S_k}{\mu_k} J_0(\mu_k R).$$

The solution for layer 1 can be reduced [10] to the finding of a biharmonic stress function $\Phi(\rho, z)$ whose values on the body surface are given by the boundary conditions. Employing the approach of [11] one seeks the solution $\Phi(\rho, z)$ in the form

$$\Phi(\rho, z) = z(A\rho^2 + Bz^2) + \sum_{k=1}^{\infty}[E_k I_0(\lambda_k\rho) + G_k\lambda_k\rho I_1(\lambda_k\rho)]\sin\lambda_k z +$$
$$+ \sum_{k=1}^{\infty}[A_k\,\mathrm{sh}\,\mu_k z + B_k\,\mathrm{ch}\,\mu_k z + C_k\mu_k z\,\mathrm{sh}\,\mu_k z + D_k\mu_k z\,\mathrm{ch}\,\mu_k z]\,J_0(\mu_k\rho),$$

where μ_k are the roots of the equation $J_1(\mu_k R) = 0$; $\lambda_k = k\pi/H$. In this case the conditions for $\rho = 0$ are automatically satisfied. By using the boundary conditions (1)-(5) and (9) one finds

$$\frac{\partial}{\partial z}\left[\nu\Delta\Phi - \frac{\partial^2\Phi}{\partial\rho^2}\right]_{\rho=R} = 0, \quad \frac{\partial}{\partial\rho}\left[(1-\nu)\,\Delta\Phi - \frac{\partial^2\Phi}{\partial z^2}\right]_{z=0} = H^*\sum_{k=1}^{\infty} S_k J_1(\mu_k\rho),$$
$$\frac{\partial}{\partial z}\left[(2-\nu)\,\Delta\Phi - \frac{\partial^2\Phi}{\partial z^2}\right]_{z=0} = 0, \quad \frac{\partial}{\partial\rho}\left[(1-\nu)\,\Delta\Phi - \frac{\partial^2\Phi}{\partial z^2}\right]_{z=H} = 0, \qquad (12)$$
$$\frac{\partial}{\partial z}\left[(2-\nu)\,\Delta\Phi - \frac{\partial^2\Phi}{\partial z^2}\right]_{z=H} = 0, \quad \frac{\partial}{\partial\rho}\left[(1-\nu)\,\Delta\Phi - \frac{\partial^2\Phi}{\partial z^2}\right]_{\rho=R} = 0.$$

To determine the unknown coefficients S_k one has additionally the condition

$$-\frac{1+\nu}{E}\frac{\partial^2\Phi}{\partial\rho\partial z}\Big|_{z=0} = \frac{1-\nu_*^2}{E_*}\sum_{k=1}^{\infty}\frac{S_k}{\mu_k^2} + \rho\left[(\alpha^* - \alpha)\,\Delta T - \frac{1-\nu_*}{E_*}\sum_{k=1}^{\infty}\frac{S_k}{\mu_k} \times J_0(\mu_k R)\right]. \qquad (13)$$

Eliminating the coefficients A, B, A_k, B_k, and E_k from (12), (13) and introducing another variable,

$$X_k = \lambda_k^4 H I_1(\lambda_k R)\,[2(1-\nu) + \varepsilon_k]\,G_k,$$

$$Y_k = \mu_k^4 R J_0(\mu_k R)\,\mathrm{sh}\,\mu_k H\left(C_k\,\mathrm{cth}\,\frac{\mu_k H}{2} + D_k\right),$$

$$Z_k = \mu_k^4 R J_0(\mu_k R)\,\mathrm{sh}\,\mu_k H\left(C_k\,\mathrm{th}\,\frac{\mu_k H}{2} + D_k\right),$$

$$T_k = \mu_k R H^* J_0(\mu_k R)\,S_k,$$

one obtains a set of infinite systems of linear equations,

$$\sum_{k=1}^{\infty}(M_{pk}T_k + N_{pk}Y_k) = X_p\ (p = 1, 3, 5, \ldots),\ \sum_{k=1}^{\infty}(M_{pk}T_k - N_{pk}Z_k) = X_p\ (p = 2, 4, 6, \ldots),$$

$$\beta_k Y_k - \sum_{p=1,3,5,\ldots} L_{pk}X_p = T_k, \quad -\gamma_k Z_k - \sum_{p=2,4,6,\ldots} Q_{pk}X_p = T_k, \tag{14}$$

$$\omega_k T_k + \sum_{p=1}^{\infty}\varphi_{pk}X_p - P_k Z_k + F_k Y_k = -\sum_{p=1}^{\infty}\delta_p T_p + (\alpha^* - \alpha)\Delta T,$$

where

$$M_{pk} = \frac{2[2(1-\nu)+\varepsilon_p]}{R^2[\varkappa_k + 2(1+\nu)/(\lambda_k R)^2](\lambda_p^2+\mu_k^2)}; \quad N_{pk} = M_{pk}\frac{2}{(\lambda_p^2+\mu_k^2)};$$

$$L_{pk} = \frac{8\mu_k^3}{H\,\mathrm{th}\frac{\mu_k H}{2}[2(1-\nu)+\varepsilon_p](\lambda_p^2+\mu_k^2)}; \quad Q_{pk} = L_{pk}\,\mathrm{th}^2\frac{\mu_k H}{2};$$

$$\varphi_{pk} = \frac{1+\nu}{E}\frac{\mu_k^2}{RH}\frac{1}{\lambda_p^2(\lambda_p^2+\mu_k^2)}; \quad \delta_p = \left(\frac{1-\nu_*}{E_* H^*} - \frac{1-\nu}{EH}\right)\frac{1}{\mu_p^2 R};$$

$$\beta_k = \frac{\mathrm{sh}\,\mu_k H - \mu_k H}{\mathrm{sh}\,\mu_k H}; \quad \gamma_k = \frac{\mathrm{sh}\,\mu_k H + \mu_k H}{\mathrm{sh}\,\mu_k H}; \quad \omega_k = \frac{1-\nu_*^2}{2E_* H^* \mu_k^2 R};$$

$$P_k = \frac{1-\nu^2}{2E}\frac{\mathrm{cth}\frac{\mu_k H}{2}}{\mu_k R}; \quad F_k = P_k\,\mathrm{th}^2\frac{\mu_k H}{2};$$

$$\varkappa_k = \frac{I_0(\lambda_k R)I_2(\lambda_k R)}{I_1^2(\lambda_k R)} - 1 + \frac{2I_2(\lambda_k R)}{\lambda_k R I_1(\lambda_k R)}; \quad \varepsilon_k = \lambda_k R\frac{I_0(\lambda_k R)}{I_1(\lambda_k R)}.$$

In finding the set of equations (14) the Fourier–Bessel expansion was used for the functions ρ, $I_0(\lambda_k\rho)$ and $\lambda_k\rho I_1(\lambda_k\rho)$ in addition to the Dini and Fourier series employed in [11]. The evaluation of the required coefficients, stress values, and displacements was carried out on an electronic computer.

To provide an example the stresses and displacements were evaluated for a silicon system (layer 1) and a silica system (layer 2) with the following values of the parameters: $E = 1.7\cdot10^4$ kg/mm²; $E_* = 0.6\cdot10^4$ kg/mm²; $\nu = 0.28$; $\nu_* = 0.17$; $(\alpha^* - \alpha)\Delta T = 4.2\cdot10^{-3}$.

In Fig. 2, $\sigma_{\rho\rho}$ is shown in layer 1 against the quantity ρ/R for the systems with the following geometrical characteristics: $H^* = 4\cdot10^{-4}$ cm; $H = 0.15$ cm; $R = 0.75$ cm (Fig. 2a); $H^* = 4\cdot10^{-4}$ cm; $H = 0.3$ cm; $R = 0.5$ cm (Fig. 2b). The values at the curves show the coordinate z, cm. It can be observed that near the center of the disk the exact solutions agree

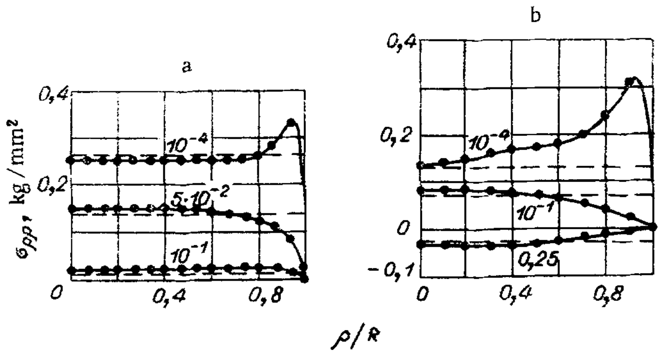

Fig. 2

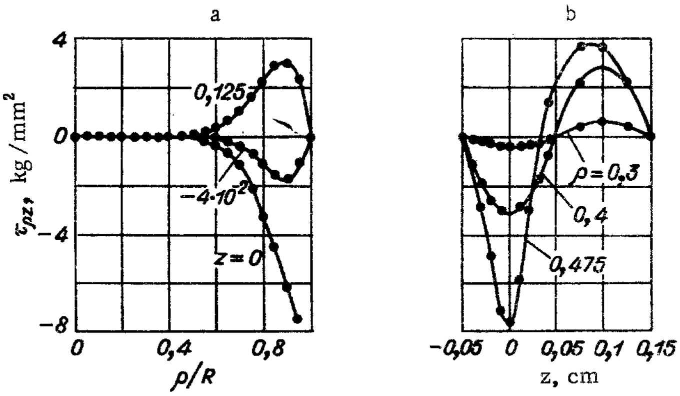

Fig. 3

sufficiently well with the computations carried out by using the "bimetallic" model (dashed curves). At the same time, near the edges of the disk the exact solution gives $\sigma_{\rho\rho}$ against ρ especially in the case of $(H + H^*)/R \sim 1$ (see Fig. 2b). The highest concentration of stresses can then be observed near the edges of the disk in the plane z = 0.

In Fig. 3 for a system with $H^* = 5\cdot10^{-2}$ cm, H = 0.15 cm, and R = 0.5 cm, $\tau_{\rho z}$ is shown versus ρ (Fig. 3a) and versus z (Fig. 3b) in layers 1 and 2 (the maximum value is $\delta \sim 0.03$). It can be seen that the tangential stresses increase in their absolute value as the edge of the disk is approached, the strongest growth being observed in the plane z = 0, as in the previous case.

Thus, in contrast to the beam approximation the solution in its exact form gives the magnitude of the internal stresses versus the coordinate ρ and enables one to compute the tangential stresses on the planes z = const.

In view of the latter the obtained solution makes it possible to predict the changes in the physical properties by the disk radius and also to estimate the demands as regards the adhesive properties of the materials in real two-layer systems.

LITERATURE CITED

1. S. P. Timoshenko, Stability of Rods, Plates, and Shells [in Russian], Nauka, Moscow (1971).
2. S. D. Brotherton, T. G.Read, D. R. Lamb, and A. F. Willoughly, "Surface charge and stress in the $Si-SiO_2$ system," Solid State Electron., 16, No. 12,1367 (1973).
3. E. I. Grigolyuk and V. M. Tolkachev "Theory of multilayer thermostat," Izv. Sib. Otd. Akad. Nauk SSSR, Ser. Tekh. Nauk, 10, No. 3, 49 (1963).
4. B. J. Aleck, "Thermal stresses in a rectangular plate fastened along its edge," J. Appl. Mech., 16, No. 2, 118 (1949).
5. R. Zeyfang, "Stresses and strains into the plate fastened on a substrate: semiconductor devices." Solid State Electron., 14, No. 10, 1035 (1975).
6. L. S. Leibenzon, A Short Course in Elasticity Theory [in Russian], TTL, Moscow–Leningrad (1942).
7. D. S. Griffin and P. B. Kellog, "Numerical solution of axisymmetric and planar elasticity problems," in: Mekhanika [Collection of Russian translations of foreign articles], No. 12 (1968), p. 111.
8. G.E. Forsythe and W. R. Wasow, Finite Difference Methods for Partial Differential Equations, Wiley–Interscience (1960).
9. G. I. Marchuk, Methods of Numerical Mathematics [in Russian], Nauka, Novosibirsk (1973).
10. A. Lyav, Mathematical Theory of Elasticity [in Russian], ONTI, Moscow (1935).
11. B. L. Abramyan, "Axisymmetric deformation of a circular cylinder," Dokl. Akad. Nauk, ArmSSR, 19 (1954).